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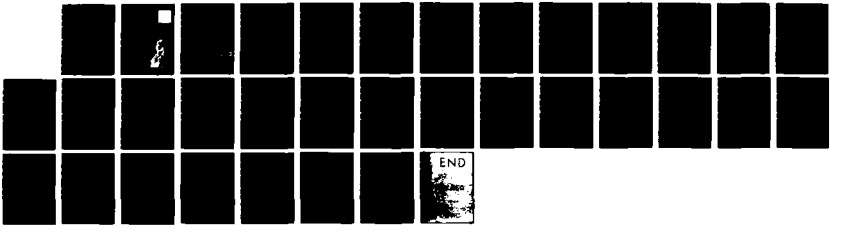

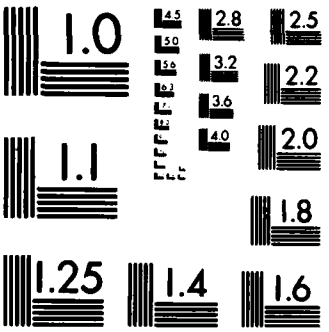

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ESTIMATING THE SCALE PARAMETER OF AN EXPONENTIAL DISTRIBUTION FROM A SAMPLE OF TIME-CENSORED r-th ORDER STATISTICS

by

S. Zacks
Department of Mathematical Sciences
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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: GWU/IMSE/Serial T-495/84	2. GOVT ACCESSION NO. AD-A148 260
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): ESTIMATING THE SCALE PARAMETER OF AN EXPONENTIAL DISTRIBUTION FROM A SAMPLE OF TIME-CENSORED r-th ORDER STATISTICS	5. TYPE OF REPORT & PERIOD COVERED: SCIENTIFIC
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): SHELEMYAHU ZACKS	8. CONTRACT OR GRANT NUMBER(s): Supported in Part by N00014-83-K-0216
9. PERFORMING ORGANIZATION NAME AND ADDRESS: GEORGE WASHINGTON UNIVERSITY, INSTITUTE FOR MANAGEMENT SCIENCE AND ENGINEERING, WASHINGTON, DC 20052	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: OFFICE OF NAVAL RESEARCH, CODE 411 S&P, ARLINGTON, VIRGINIA 22217	12. REPORT DATE: 9 October 1984
	13. NUMBER OF PAGES: 29
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC SALE AND RELEASE; DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

EXPONENTIAL DISTRIBUTION
RELIABILITY
ESTIMATORS
MOMENT-EQUATION ESTIMATORS
BIOASSAY
TIME-CENSORED ORDER STATISTIC
MAXIMUM LIKELIHOOD

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The problem of estimating the mean, θ, of an exponential distribution is studied, when the data available are a random sample of time-censored r-th order statistics. Examples for such empirical situations are cited in the paper. Maximum Likelihood (MLE) and Moment Equation (MEE) estimators are studied. Theoretical derivations are provided for the large sample variances and distributions of these estimators. The efficiency of the MEE compared to the MLE is studied.

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GWU/IMSE/Serial T-495/84
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The problem of estimating the mean, θ , of an exponential distribution is studied, when the data available are a random sample of time-censored r-th order statistics. Examples for such empirical situations are cited in the paper. Maximum Likelihood (MLE) and Moment Equation (MEE) estimators are studied. Theoretical derivations are provided for the large sample variances and distributions of these estimators. The efficiency of the MEE compared to the MLE is studied.

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0. Introduction

Methods of estimating the scale parameter of exponential life distributions are generally based on complete information of the failure times, with or without censoring. More specifically, the estimation is based on life testing of n identical systems which are subjected to similar experimental conditions. The failure times of the systems are recorded, and the experiment is censored either at the r-th failure $(1 < r \leq n)$, which is called censoring of Type II, or at a fixed time, t_o, which is censoring of type I. In some cases the minimum of t_o and the r-th failure is used. The reader is referred to any one of the text-books on statistical methods in reliability for the details of the estimation procedures (see for example W. Nelson [5]). The present

paper has been motivated by two papers of Epstein et al [2,3], which discuss photodynamic bioassays for estimating the potency of benzo-solubale organic extracts from particulates in air samples. In these experiments various dosages of organic extracts and of their derived fractions (basic, acidic, aliphatic, aromatic, oxygenated and water soluble) were applied on $n = 30$ living organisms (Paramicium coudatum) under ultraviolet irradiation. In each application the measured biological response was the time required (in minutes) to immobilize 90% of the cells. This response variable was called the lethal time 90 (LT90). Each trial was censored at $t_o = 90$ minutes, and the number of cells still living at the time of censoring was not recorded. Empirical fitting of a dosage-response curve was done by Epstein et al [2]. Serious questions arise concerning the validity of such an empirical analysis, which is performed without sufficient theoretical justification. The present paper is devoted to a theoretical analysis of two possible estimators of a scale parameter, when the data are the censored r-th order statistic. More specifically, let $X_{n,r}$ denote the r-th order statistic of a sample of n i.i.d. random variables from an exponential distribution with mean θ. Let $X^*_{n,r} = \min(X_{n,r}, t_o)$. A random sample of m observations on $X^*_{n,r}$ is available. In Section 1 we present the model and define some basic notions. In addition we provide formulae of the expected value and variance of $X^*_{n,r}$. On the basis of these formulae we study in Section 2 the structure and properties of a moment-equation estimator (MEE) of θ. This is the value of θ for which $E_\theta\{X^*_{n,r}\}$ is equal to the sample mean of the m observations on $X^*_{n,r}$. We show that if at least one observation is not censored then the MEE is unique. We further show that as m grows, the asymptotic distribution of the MEE is normal. Formulae for the asymptotic bias and asymptotic variance of the MEE are also derived. In Section 3 we study the structure of the maximum likelihood estimator (MLE) of θ and show that it is unique, provided at least one observation is uncensored. The Fisher Information Function and the asymptotic normal distribution of the MLE are derived

in Section 4. In Section 5 we discuss the asymptotic relative efficiency of the MEE compared to the MLE, and provide some simulation results which illustrate the performance of these estimators in samples of size $m = 20(10)50$. The results of the present study can be applied not only to the analysis of the photodynamic bioassays data of Epstein et al [2,3] but also to reliability studies, in which the observed failure times of systems are based on censored r-th order statistics. This is the case, for example, when n identical components within a system function independently, and the system fails when the r-th component failures happen. The times of the component failures are not available.

1. The Model and Moments of the Censored Observations

Consider a sequence of independent replicas of an experiment in which n systems are subjected t_o life testing (possibly accelerated). Each experiment is terminated either at the r-th failure $(1 < r < n)$, or at a fixed time point, t_o, whichever comes first. The recorded random variable is the time-censored r-th order statistic. We further assume that the life distribution of the individual systems is an exponential distribution, i.e., $F(x/\theta) = 1 - \exp\{-x/\theta\}$, $0 < x < \infty$ and $0 < \theta < \infty$. Let $X_{n,r}$ denote the r-th order statistic in a sample of n i.i.d. random variables, having a common distribution $F(\frac{x}{\theta})$. The observed random variable in the time-censored experiment is $X^* = \min(t_o, X_{n,r})$. Since the life-time distribution $F(\frac{x}{\theta})$ is exponential, the p.d.f. of $X_{n,r}$ is (see David [1])

$$(1.1) \quad f_{n,r}(x;\theta) = r\binom{n}{r} \frac{1}{\theta} \exp\{-(n-r+1)\frac{x}{\theta}\} (1-\exp\{-\frac{x}{\theta}\})^{r-1},$$

for $0 < x < \infty$. Let $U_{n,r} = X_{n,r}/\theta$ be the standardized r-th order statistic, and let $f_{n,r}(u)$ denote the standard density, i.e., $f_{n,r}(x;\theta) = \frac{1}{\theta} f_{n,r}(\frac{x}{\theta})$. Let $G_{n,r}(x; \theta, t_o)$ designate the c.d.f. of X^*. This c.d.f. is absolutely continuous on $[0,t_o)$ and has a jump point at t_o, i.e.,

$$(1.2)\quad G_{n,r}(x;\theta,t_o)=\begin{cases}0 & ,\ x\le 0\\ \int_0^x f_{n,r}(x';\theta)\,dx' & ,\ 0<x<t_o\\ 1 & ,\ x\ge t_o\ .\end{cases}$$

The height of the jump of this c.d.f. at $x = t_o$ is

$$(1.3)\quad P_\theta\{X_{n,r}\ge t_o\}=B(r-1;\,n,\,F(\tfrac{t_o}{\theta}))\ ,$$

where $B(j;\,n,\,p)$ designates the c.d.f. of the Binomial distribution with parameters $(n,\,p)$. We derive now formulae for the expected value and variance of the censored variable X^*. It is well known (see David [1]) that if $t_o=\infty$, (no censoring) then $E\{X^*|\theta,\,t_o=\infty\}=\theta\sum_{j=o}^{r-1}\frac{1}{n-j}$ and $V\{X^*|\theta,\,t_o=\infty\}=\theta^2\sum_{j=o}^{r-1}\frac{1}{(n-j)^2}$. This is due to the fact that the r-th order statistic in the exponential case can be represented as a sum of r independent random variables, having exponential distributions with means $\theta/(n-j)$, $j=0,\ldots,r-1$. In the time censored case one obtains

Lemma 1.1. The first two moments of X^* in the time censored case are:

$$(1.4)\quad E\{X^*|\theta,\,t_o\}=\theta\sum_{j=o}^{r-1}\frac{1}{n-j}\left[1-B\left(j;\,n,\,F(\tfrac{t_o}{\theta})\right)\right]$$

and

$$(1.5)\quad E\{X^{*2}|\theta,\,t_o\}=2\theta^2\left\{\sum_{j=o}^{r-1}\frac{1}{n-j}\sum_{k=o}^{j}\frac{1}{n-k}\left[1-B(k;n,F(\tfrac{t_o}{\theta}))\right]-\frac{t_o}{\theta}\cdot\sum_{j=o}^{r-1}\frac{1}{n-j}\,B(j;\,n,\,F(\tfrac{t_o}{\theta}))\right\}$$

Proof:

According to (1.2) ,

$$(1.6)\quad E\{X^{*}|\theta,\ t_o\} = \int_o^{t_o} x\ f_{n,r}(x;\theta)\ dx + t_o\ B(r-1;\ n,\ F(\frac{t_o}{\theta}))\ ,$$

and

$$(1.7)\quad E\{X^{*2}|\theta,\ t_o\} = \int_o^{t_o} x^2\ f_{n,r}(x;\theta)\ dx + t_o^2\ B(r-1;\ n,\ F(\frac{t_o}{\theta}))\ .$$

Let $\eta = t_o/\theta$ then

$$(1.8)\quad \int_o^{t_o} x\ f_{n,r}(x;\theta)\ dx = \theta \int_o^{\eta} u f_{n,r}(u)\ du \equiv \theta\ \mu_{n,r}^{(1)}(\eta)\ .$$

Moreover, by interchanging the order of integration we obtain

$$(1.9)\quad \mu_{n,r}^{(1)}(\eta) = \int_o^{\eta} (\int_o^{u} dt)\ f_{n,r}(u)\ du$$

$$= \int_o^{\eta} (\int_t^{\eta} f_{n,r}(u)\ du)\ dt$$

$$= \int_o^{\eta} [B(r-1;\ n,\ F(\frac{t}{\theta})) - B(r-1;\ n,\ F(\frac{t_o}{\theta}))]\ dt$$

$$= \int_o^{\eta} B(r-1;\ n,\ 1-e^{-t})\ dt - \eta\ B(r-1;\ n,\ 1-e^{-\eta})\ .$$

Furthermore,

$$(1.10)\quad \int_o^{\eta} B(r-1;\ n,\ 1-e^{-t})\ dt = \sum_{j=o}^{r-1} \binom{n}{j} \int_o^{\eta} (1-e^{-t})^j\ e^{-(n-j)t} dt$$

$$= \sum_{j=o}^{r-1} \binom{n}{j} \frac{1}{(j+1)\binom{n}{j+1}} \int_o^{\eta} f_{n,j+1}(t)\, dt$$

$$= \sum_{j=o}^{r-1} \frac{1}{n-j} \left[1 - B(j;\, n,\, 1-e^{-\eta})\right] .$$

Finally, substituting (1.9) and (1.10) in (1.6) we obtain (1.4). Notice that $\lim_{\eta \to \infty} B(j;\, n,\, i-e^{-\eta}) = 0$ for each $j < n$. Hence, the well known formula of $E\{X^* | \theta,\, t = \infty\}$ is a limiting case of (1.4). In a similar fashion, we derive the formula

$$(1.11)\quad \mu^{(2)}_{n,r}(\eta) = \int_o^{\eta} u^2 f_{n,r}(u)\, du$$

$$= 2 \int_o^{\eta} \left(\int_o^{u} t\,dt\right) f_{n,r}(u)\, du$$

$$= 2 \int_o^{\eta} t \left(\int_t^{\eta} f_{n,r}(u)\,du\right) dt$$

$$= 2 \int_o^{\eta} t\, B(r-1;\, n,\, 1-e^{-t})\,dt - \eta^2 B(r-1;\, n,\, 1-e^{-\eta})$$

Furthermore,

$$(1.12)\quad \int_o^{\eta} t\, B(r-1;\, n,\, 1-e^{-t})\,dt = \sum_{j=o}^{r-1} \binom{n}{j} \int_o^{\eta} t\, (1-e^{-t})^j\, e^{-t(n-j)}\, dt$$

$$= \sum_{j=o}^{r-1} \frac{1}{n-j} \int_o^{\eta} t\, f_{n,j+1}(t)\,dt$$

$$= \sum_{j=o}^{r-1} \frac{1}{n-j} \left[\sum_{k=o}^{j} \frac{1}{n-k} \left(1-B(k;\, n,\, 1-e^{-\eta})\right) - \eta\, B(j;\, n,\, 1-e^{-\eta}) \right] .$$

Substituting (1.11) and (1.12) in (1.7) with the proper scale factor, we obtain formula (1.5).

(Q.E.D.)

From formulae (1.4) and (1.5) one obtains immediately the following formula for the variance of $X^*_{n,r}$, namely:

$$(1.13)\quad \sigma^2(\theta;\, t_o) = V\{X^*_{n,r};\, \theta,\, t_o\}$$

$$= \theta^2 \left\{ 2 \sum_{j=o}^{r-1} \frac{1}{n-j} \sum_{k=o}^{j} \frac{1}{n-k} \left[1-B(k;\, n,\, F(\tfrac{t_o}{\theta}))\right] \right.$$

$$- 2\,\frac{t_o}{\theta} \sum_{j=o}^{r-1} \frac{1}{n-j}\, B(j;\, n,\, F(\tfrac{t_o}{\theta})) -$$

$$\left. \left(\sum_{j=o}^{r-1} \frac{1}{n-j} \left(1-B(j;\, n,\, F(\tfrac{t_o}{\theta}))\right) \right)^2 \right\} .$$

One can easily check that $\lim_{t_o \to \infty} \sigma^2(\theta;\, t_o) = \theta^2 \sum_{j=o}^{r-1} \frac{1}{(n-j)^2}$.

In the following section we apply the above formulae for the study of the moment-equation estimator, MEE.

2. The Moment Equation Estimator and its Asymptotic Properties

In the case of no time censoring, i.e., when $t_o = \infty$, an unbiased estimator of θ is obtained by equating the sample mean $\bar{X}_m = \frac{1}{m} \sum_{i=1}^{m} X^{(i)}_{n,r}$ to $E\{X^* | \theta,\, t_o = \infty\}$ and solving for θ. In other words, if $t_o = \infty$ then

$$(2.1)\quad \hat{\theta}(\bar{X}_m) = \bar{X}_m \Big/ \sum_{j=o}^{r-1} \frac{1}{n-j}$$

is an unbiased estimator of θ, having a variance

$$(2.2)\quad V\{\hat{\theta}(\bar{X}_m)|\theta,\ t_o = \infty\} = \theta^2 \sum_{j=o}^{r-1} \frac{1}{(n-j)^2} \Big/ m\left(\sum_{j=o}^{r-1} \frac{1}{n-j}\right)^2 .$$

In the present section we generalize this estimator for the case of $0 < t_o < \infty$ and determine its asymptotic properties. Since t_o is fixed, we simplify the notation by letting $E(\theta) \equiv E\{X^*|\theta,\ t_o\}$. Since the sample represents m i.i.d. random variables $X_1^*,\ldots,X_m^*$, the sample mean $\bar{X}_m^* = \frac{1}{m}\sum_{j=1}^{m} X_j^*$ is an unbiased, strongly consistent estimator of $E(\theta)$. Accordingly we say that $\hat{\theta}$ $(\bar{X}_m^*)$ is a moment equation estimator (MEE) of θ if it is the root of the equation

$$(2.3)\quad \theta \sum_{j=o}^{r-1} \frac{1}{n-j}\ [1-B(j;\ n,\ 1-e^{-t_o/\theta})] = \bar{X}_m^* .$$

The left-hand side of (2.3) is the function $E(\theta)$, which is defined on the domain $(0,\infty)$.

Lemma 2.1

(i) $E(\theta)$ is a strictly increasing function on $(0,\infty)$;

(ii) $E(\theta)$ is concave on $(1,\infty)$;

(iii) $\lim_{\theta\to\infty} E(\theta) = t_o$.

Proof:

(i) straightforward differentiation yields

$$(2.4)\quad \frac{d}{dp} B(j;\ n,\ p) = \frac{-(n-j)}{1-p}\ b(j;\ n,\ p) \qquad ,\ j=0,\ldots,n$$

and every $0 < p < 1$, where $b(j;\ n,\ p)$ is the Binomial p.d.f., i.e., $b(j;\ n,\ p) = B(j;\ n,\ p) - B(j-1;\ n,\ p)$.

Thus,

$$(2.5)\quad \frac{d}{d\theta} E(\theta) = \sum_{j=o}^{r-1} \frac{1}{n-j} [1-B(j;\, n,\, 1-e^{-t_o/\theta})]$$

$$- \frac{t_o}{\theta} \sum_{j=o}^{r-1} b(j;\, n,\, 1-e^{-t_o/\theta})$$

$$= \sum_{j=o}^{r-1} \frac{1}{n-j} [1-B(j;\, n,\, 1-e^{-t_o/\theta})] - \frac{t_o}{\theta} B(r-1;\, n,\, 1-e^{-t_o/\theta}) .$$

Comparing (2.5) with (1.9) - (1.10) we obtain that

$$(2.6)\quad \frac{d}{d\theta} E(\theta) = \mu_1(\frac{t_o}{\theta}) > 0 \quad , \text{ for all } \theta \text{ in } (0,\infty) .$$

This proves that $E(\theta)$ is a strictly increasing function over $(0,\infty)$.

(ii) Differentiating (2.5) we obtain

$$(2.7)\quad \frac{d^2}{d\theta^2} E(\theta) = -\frac{t_o}{\theta} B(r-1;\, n,\, 1-e^{-t_o/\theta}) + \frac{t_o}{\theta^2} B(r-1;\, n,\, 1-e^{-t_o/\theta})$$

$$- \frac{(n-r+1)t_o^2}{\theta^3} b(r-1;\, n,\, 1-e^{-t_o/\theta}) .$$

Thus, for every $\theta \geq 1$, $\frac{d^2}{d\theta^2} E(\theta) < 0$, which proves the concavity of $E(\theta)$ over the interval $(1,\infty)$.

$$(2.8)\quad \lim_{\theta\to\infty} E(\theta) = \lim_{\theta\to\infty} \theta \sum_{j=o}^{r-1} \frac{1}{n-j} [1-B(j;\, n,\, 1-e^{-t_o/\theta})]$$

Starting with $j = 0$ we obtain

$$(2.9)\quad \lim_{\theta\to\infty} \theta\, [1-B(0;\, n,\, 1-e^{-t_o/\theta})] =$$

$$\lim_{\theta\to\infty} \theta(1-e^{-nt_o/\theta}) = nt_o .$$

On the other hand, for every $j \geq 1$,

$$(2.10)\quad \lim_{\theta\to\infty} \theta[1-B(j;\, n,\, 1-e^{-t_o/\theta})] = \lim_{\theta\to\infty} \theta(1-e^{-nt_o/\theta}) - \lim_{\theta\to\infty} \sum_{i=1}^{j} \binom{n}{i} (1-e^{-t_o/\theta})^i e^{-(n-i)t_o/\theta} = 0 .$$

Indeed,

$$(2.11)\quad \lim_{\theta\to\infty} \theta(1-e^{-t_o/\theta})^i e^{-(n-i)t_o/\theta} = \begin{cases} t_o & , \text{ if } i=1 \\ 0 & , \text{ if } i \geq 1 \end{cases}$$

Thus, from (2.8), (2.9) and (2.10) we obtain the result.

(Q.E.D.)

Corollaries:

(i) There exists a unique solution to equation

(2.3) , $\hat{\theta}(\bar{X}_m^*)$, for each $0 < \bar{X}_m^* < t_o$.

(ii) $\hat{\theta}(\bar{X}_m^*)$ does not exist if $\bar{X}_m^* = t_o$.

The solution of equation (2.3) can follow the Newton-Raphson iterative procedure

$$(2.12)\quad \hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - \frac{E(\hat{\theta}^{(i)}) - \bar{X}_m^*}{E'(\hat{\theta}^{(i)})} \quad , \quad i=1,2,\ldots$$

where $\hat{\theta}^{(1)}$ is any initial solution, and $E'(\theta)$ is the derivative of $E(\theta)$, given by (2.5). If M_s denotes the number of censored observations in a sample of m , then $\hat{P}_s = M_s/m$ is a strongly consistent estimator of $B(r-1;\, n,\, 1-e^{-t_o/\theta})$, as $m\to\infty$. Thus, a solution, θ_s , of

$$(2.13)\quad \hat{P}_s = B(r-1;\ n,\ 1-e^{-t_o/\theta})$$

is a consistent estimator of θ and could serve as an initial value $\hat{\theta}^{(1)}$ in the sequence defined by (2.12). If $M_s = 0$ we can start with the "unbiased" estimator $\hat{\theta}^{(1)} = \bar{X}_m^* / \sum_{j=o}^{r-1} \frac{1}{n-j}$.

We derive now formulae for the asymptotic mean and variance of $\hat{\theta}(\bar{X}_m^*)$, as $m \to \infty$.

According to the Central Limit Theorem, $\sqrt{m}\,(\bar{X}_m^* - E(\theta)) \xrightarrow{d} N(0, \sigma^2(\theta; t_o))$. Let $E^{-1}(x)$ denote the unique inverse of $E(\theta)$. Notice that the MEE is the value of $E^{-1}(\bar{X}_m^*)$. Moreover, since $E^{-1}(E\{\bar{X}_m^*;\ \theta,\ t_o\}) = E^{-1}(E(\theta)) = \theta$, we consider the Taylor expansion of $E^{-1}(x)$ around $E(\theta)$ and obtain

$$(2.14)\quad \hat{\theta}(\bar{X}_m^*) = \theta + (\bar{X}_m^* - E(\theta)) \frac{1}{E'(\theta)} + o_p\left(\frac{1}{\sqrt{m}}\right), \quad \text{as } m \to \infty.$$

Thus,

$$(2.15)\quad \sqrt{m}\ (\hat{\theta}(\bar{X}_m^*) - \theta) \xrightarrow{d} N\left(0, \frac{\sigma^2(\theta; t_o)}{(E'(\theta))^2}\right).$$

Moreover, since $\bar{X}_m \xrightarrow{a.s.} E(\theta)$, and since $E^{-1}(x)$ is a continuous transformation, $\hat{\theta}(\bar{X}_m^*)$ is a strongly consistent estimator of θ, as $m \to \infty$. The asymptotic variance of $\hat{\theta}(\bar{X}_m^*)$ is thus,

$$(2.16)\quad AV\{\hat{\theta}(\bar{X}_m^*);\ \theta,\ t_o\} = \frac{\sigma^2(\theta; t_o)}{m(E'(\theta))^2}$$

This asymptotic variance will be compared later with the asymptotic variance of the MLE, to determine the asymptotic relative efficiency of the MEE.

In order to obtain an expression for the asymptotic bias of the MEE, we add another term to the expansion (2.14).

The second order derivative of $E^{-1}(x)$ is

$$(2.17)\quad \frac{d}{dx}\,\frac{1}{E'(E^{-1}(x))} = -\,\frac{E''(E^{-1}(x))}{(E'(E^{-1}(x)))^3}$$

Hence, we obtain the expansion

$$(2.18)\quad \hat{\theta}(\bar{X}_m^*) = \theta + (\bar{X}_m^* - \theta)\,\frac{1}{E'(\theta)} - \frac{1}{2}\,(\bar{X}_m^* - \theta)^2\,\frac{E''(\theta)}{(E'(\theta))^3} + o_p\left(\frac{1}{m}\right),$$

as $m \to \infty$. It follows that the asymptotic bias of $\hat{\theta}(\bar{X}_m^*)$ is

$$(2.19)\quad B(\theta;\, t_o) = \frac{-\sigma^2(\theta;\, t_o)}{2m} \cdot \frac{E''(\theta)}{(E'(\theta))^3}$$

According to Lemma 2.1 (ii), $B(\theta;\, t_o) > 0$ for all $\theta \geq 1$.

3. Maximum Likelihood Estimation

Let $\underset{\sim}{X}_m^* = (X_1^*, \ldots, X_m^*)$ be a vector of m i.i.d. random variables. The likelihood function of θ based on $\underset{\sim}{X}_m^*$, up to a factor of proportionality, is given by

$$(3.1)\quad L(\theta;\, \underset{\sim}{X}_m^*) = \prod_{i=1}^{m} [f_{n,r}(X_i^*;\theta)]^{I_i}\, [B(r-1;\, n,\, F(\tfrac{t_o}{\theta}))]^{1-I_i}$$

where

$$(3.2)\quad I_i = \begin{cases} 1 & , \text{ if } X_i^* < t_o \\ 0 & , \text{ if } X_i^* = t_o \end{cases}.$$

Let $K_m = \sum_{i=1}^{m} I_i$ and let $\ell(\theta;\, \underset{\sim}{X}_m^*)$ denote the log-likelihood

function, then

$$(3.3)\quad \ell(\theta; \underset{\sim m}{X^*}) = - K_m \log \theta + (m-K_m) \log B(r-1;\, n,\, F(\tfrac{t_o}{\theta})) + \sum_{i=1}^{m} I_i \left[(r-1) \log (1-\exp\{- X_i^*/\theta\}) - (n-r+1)\, X_i^*/\theta\right]$$

The maximum likelihood estimator of θ (MLE) is the value of θ in $(0,\infty)$ which maximizes $\ell(\theta; \underset{\sim m}{X^*})$.

Consider the score-function, $S(\theta; \underset{\sim m}{X^*})$, which is the partial derivative of $\ell(\theta; \underset{\sim m}{X^*})$, with respect to θ . According to (2.4) we obtain that

$$(3.4)\quad S(\theta; \underset{\sim m}{X^*}) = - \frac{K_m}{\theta} + (m-K_m)\,(n-r+1)\,\frac{t_o}{\theta^2} \cdot \frac{b(r-1;\, n,\, 1-e^{-t_o/\theta})}{B(r-1;\, n,\, 1-e^{-t_o/\theta})} + \sum_{i=1}^{m} I_i \left[\frac{(n-r+1)X_i^*}{\theta^2} - \frac{(r-1)}{\theta^2} \cdot \frac{X_i^*\, e^{-X_i^*/\theta}}{1-\exp\{-X_i^*/\theta\}}\right]$$

Lemma 3.1. If $K_m = 0$ then there is no MLE of θ .

Proof: If $I_i = 0$ for all i then, according to (3.4) $S(\theta; \underset{\sim m}{X^*}) > 0$ for all θ in $(0,\infty)$.

Theorem 3.1. If $K_m \geq 1$ then there exists a unique MLE.

Proof:

The equation $S(\theta; \underset{\sim}{X}^*_m) \equiv 0$ is equivalent to the equation

$$(3.5)\quad \theta = \frac{n}{K_m} \sum_{i=1}^{m} I_i X_i^* - \frac{r-1}{K_m} \sum_{i=1}^{m} \frac{I_i X_i^*}{1-e^{-X_i^*/\theta}} + \frac{t_o}{K_m} (m-K_m)(n-r+1) \frac{b(r-1; n, 1-e^{-t_o/\theta})}{B(r-1; n, 1-e^{-t_o/\theta})} .$$

Let $H(\theta; \underset{\sim}{X}^*_m)$ denote the R.H.S. of (3.5). We show first that, for fixed $\underset{\sim}{X}^*_m$,

$$(i)\quad \lim_{\theta \to o} H(\theta; \underset{\sim}{X}^*_m) = \frac{n-r+1}{K_m} \left[\sum_{i=1}^{m} I_i X_i^* + t_o (m-K_m) \right]$$

and

$$(ii)\quad \lim_{\theta \to \infty} H(\theta; X_m^*) = -\infty .$$

Indeed,

$$(3.6)\quad \frac{b(r-1; n, 1-e^{-t_o/\theta})}{B(r-1; n, 1-e^{-t_o/\theta})} = \left[1 + \sum_{j=o}^{r-2} \frac{\binom{n}{j}}{\binom{n}{r-1}} \left(e^{t_o/\theta} - 1 \right)^{-(r-1-j)} \right]^{-1}$$

Thus,

$$\lim_{\theta \to o} \frac{b(r-1; n, 1-e^{-t_o/\theta})}{B(r-1; n, 1-e^{-t_o/\theta})} = 1$$

$$(3.7)\quad \lim_{\theta \to \infty} \frac{b(r-1; n, 1-e^{-t_o/\theta})}{B(r-1; n, 1-e^{-t_o/\theta})} = 0$$

From (3.5) and (3.7) we obtain (i) and (ii). We show now that $\frac{\partial}{\partial\theta} H(\theta; \underset{\sim}{X}^*_m) < 0$ for all θ in $(0,\infty)$. This will prove that there is a unique finite θ satisfying (3.5).

Indeed,

$$(3.8)\quad \frac{\partial}{\partial\theta} H(\theta; \underset{\sim}{X}^*_m) = -\frac{r-1}{K_m \theta^2} \sum_{i=1}^{m} I_i \frac{X_i^{*2} e^{-X_i^*/\theta}}{(1-e^{-X_i^*/\theta})^2}$$

$$+ \frac{t_o}{K_m} (m-K_m)(n-r+1) \frac{\partial}{\partial\theta} \left[1 + \sum_{j=o}^{r-2} \frac{\binom{n}{j}}{\binom{n}{r-1}} \left(e^{t_o/\theta}-1\right)^{-r-1-j)}\right]^{-1} .$$

But

$$(3.9)\quad \frac{\partial}{\partial\theta} \left[1 + \sum_{j=o}^{r-2} \frac{\binom{n}{j}}{\binom{n}{r-1}} \left(e^{t_o/\theta}-1\right)^{-(r-1-j)}\right]^{-1} =$$

$$- \frac{t_o}{\theta^2} e^{t_o/\theta} \left[1 + \sum_{j=o}^{r-2} \frac{\binom{n}{j}}{\binom{n}{r-1}} \left(e^{t_o/\theta}-1\right)^{-(r-1-j)}\right]^{-2}$$

$$\cdot \sum_{j=o}^{r-2} \frac{\binom{n}{j}}{\binom{n}{r-1}} (r-1-j) \left(e^{t_o/\theta}-1\right)^{-(r-j)}$$

substituting (3.9) in (3.8) we obtain that $\frac{\partial}{\partial\theta} H(\theta; \underset{\sim}{X}^*_m) < 0$ for all θ in $(0,\infty)$.

(Q.E.D.)

Numerical examples have illustrated that the function $H(\theta; \underset{\sim}{X}^*_m)$ might be very steep in the neighborhood of θ, and the Newton-Raphson method has proven to be instable. We have used therefore a search procedure for the maximum of the log-likelihood function, which provided good numerical results.

4. The Fisher Information Function and the Asymptotic Distribution of the MLE

The time-censored r-th order statistic X^* has a c.d.f. $G_{n,r}(y; \theta, t_o)$, which is specified in (1.2). The corresponding generalized p.d.f. is

$$(4.1)\quad g_{n,r}(y; \theta, t_o) = \begin{cases} \frac{1}{\theta} f_{n,r}\left(\frac{y}{\theta}\right) & , 0 \leq y < t_o \\ B(r-1; n, F(\frac{t_o}{\theta})) & , y = t_o \end{cases}$$

We observe that these p.d.f's have the following properties:

(i) The support of $g_{n,r}(u; \theta, t_o)$ is $[0, t_o]$, independently of θ.

(ii) $g_{n,r}(y; \theta, t_o)$ has continuous partial derivatives, with respect to θ, at every $y \in [0, t_o]$. Moreover, since their support is independent of θ, $\frac{\partial^i}{\partial \theta^i} g_{n,r}(y; \theta, t_o)$ is a uniformly integrable function of y, for every $i=1,2,\ldots$ and all θ.

(iii) The p.d.f's satisfy:

$$(4.2)\quad \frac{\partial}{\partial \theta} \int_0^{t_o} d\, G_{n,r}(y; \theta, t_o) = \int_0^{t_o} \frac{\partial}{\partial \theta} \{dG_{n,r}(y; \theta, t_o)\} = 0$$

for all $\theta \in (0,\infty)$. This is proven by the following

Lemma 4.1

$$(4.3)\quad E_\theta\left\{\frac{\partial}{\partial \theta} \log g_{n,r}(X^*; \theta, t_o)\right\} = 0 \quad , \text{ all } \theta \in (0,\infty).$$

Proof:

As in (3.3) we obtain

$$(4.4)\quad \frac{\partial}{\partial\theta}\log g_{n,r}(y;\,\theta,\,t_o) =$$

$$= \begin{cases} -\frac{1}{\theta} - \frac{r-1}{\theta^2}\,\frac{ye^{-y/\theta}}{1-e^{-y/\theta}} + \frac{n-r+1}{\theta^2}\,y & ,\ 0 \le y < t_o \\ \frac{t_o}{\theta^2}(n-r+1)\,\frac{b(r-1;\,n,\,1-e^{-t_o/\theta})}{B(r-1;\,n,\,1-e^{-t_o/\theta})} & ,\ y = t_o\,. \end{cases}$$

Thus,

$$(4.5)\quad E_\theta\{\frac{\partial}{\partial\theta}\log g_{n,r}(X^*;\,\theta,\,t_o)\} =$$

$$-\frac{1}{\theta}\left[1-B(r-1;\,n,\,1-e^{-t_o/\theta})\right] + \frac{t_o}{\theta^2}(n-r+1)\,b(r-1;\,n,\,1-e^{-t_o/\theta})$$

$$-\frac{n-r+1}{\theta^2}\int_o^{t_o} y\,f_{n,r-1}(y;\theta)dy + \frac{n-r+1}{\theta^2}\int_o^{t_o} y\,f_{n,r}(y,\theta)dy\,.$$

Furthermore, according to (1.9),

$$(4.6)\quad E_\theta\left\{\frac{\partial}{\partial\theta}\log g_{n,r}(X^*;\,\theta,\,t_o)\right\} =$$

$$-\frac{1}{\theta}\left[1-B(r-1;\,n,\,1-e^{-t_o/\theta})\right] + \frac{t_o}{\theta^2}(n-r+1)\,b(r-1;\,n,\,1-e^{-t_o/\theta})$$

$$+\frac{n-r+1}{\theta^2}\left[\int_o^{t_o} b(r-1;\,n,\,1-e^{-x/\theta})\,dx - t_o\,b(r-1;\,n,\,1-e^{-t_o/\theta})\right]$$

Finally,

$$(4.7)\quad \int_o^{t_o} b(r-1;\,n,\,1-e^{-x/\theta})\,dx = \binom{n}{r-1}\int_o^{t_o}(1-e^{-x/\theta})^{r-1}\cdot e^{-(n-r+1)x/\theta}\,dx = \frac{\theta}{n-r+1}\int_o^{t_o} f_{n,r}(x;\theta)\,dx$$

$$= \frac{\theta}{n-r+1}\left[1-B(r-1;\,n,\,1-e^{-t_o/\theta})\right]\,.$$

Substituting (4.7) into (4.6) we obtain (4.3).

(Q.E.D.)

According to Lemma 4.1, the Fisher Information Function is

$$(4.8)\quad I(\theta;\, t_o) = E_o\,\{[\,\frac{\partial}{\partial\theta}\log g_{n,r}(X^*;\,\theta,\,t_o)]^2\}\ .$$

Equivalently,

$$(4.9)\quad I(\theta;\, t_o) = Q(\frac{t_o}{\theta})/\theta^2$$

where

$$(4.10)\quad Q(\eta) = \int_o^{\eta} f_{n,r}(y)\ [1+(r-1)\ \frac{e^{-y}y}{1-e^{-y}}\ -$$

$$(n-r+1)\ y]^2\ dy + \eta^2(n-r+1)^2\cdot\frac{b^2(r-1;\ n,\ 1-e^{-\eta})}{B(r-1;\ n,\ 1-e^{-\eta})}$$

Expansion of the quadratic form in (4.10) and some algebraic manipulations yield the formula

$$(4.11)\quad Q(\eta) = -\ [1-B(r-1;\ n,\ 1-e^{-\eta})]$$

$$+\ 2\eta\ (n-r+1)\ b(r-1;\ n,\ 1-e^{-\eta})$$

$$+\ \eta^2(n-r+1)^2\ b^2(r-1;\ n,\ 1-e^{-\eta})/B(r-1;\ n,\ 1-e^{-\eta})$$

$$+\ n\ \frac{n-r+1}{r-2}\ \mu^{(2)}_{n,r-2}(\eta)$$

$$+\ (n-r+1)^2\ \Delta_r^2\ \mu^{(2)}_{n,r}(\eta)\quad ,$$

where the second incomplete moment $\mu^{(2)}_{n,j}(\eta)$ is given in formula (1.11) and $\Delta_r^2\ \mu^{(2)}_{n,r}(\eta) = \mu^{(2)}_{n,r}(\eta) - 2\,\mu^{(2)}_{n,r-1}(\eta) + \mu^{(2)}_{n,r-2}(\eta)$.

It is interesting to study the effect of the time-censoring on the Fisher Information Function. For this purpose we prove that when $t_o = \infty$ then

$$(4.12) \quad I(\theta) = I(\theta;\infty) = Q/\theta^2 \quad ,$$

where

$$(4.13) \quad Q = \int_o^\infty f_{n,r}(y) \left[1+(r-1)\frac{ye^{-y}}{1-e^{-y}} - (n-r+1)y\right]^2 dy$$

$$= 1 + (r-1)^2 \int_o^\infty y^2 \frac{e^{-2y}}{(1-e^{-y})^2} f_{n,r}(y)\, dy$$

$$+ (n-r+1)^2 \int_o^\infty y^2 f_{n,r}(y)\, dy$$

$$+ 2(r-1) \int_o^\infty y \frac{e^{-y}}{1-e^{-y}} f_{n,r}(y)\, dy$$

$$- 2(n-r+1) \int_o^\infty y\, f_{n,r}(y)\, dy$$

$$- 2(r-1)(n-r+1) \int_o^\infty y^2 \frac{e^{-y}}{1-e^{-y}} f_{n,r}(y)\, dy \, .$$

Substituting

$$(4.14) \quad \frac{e^{-y}}{1-e^{-y}} f_{n,r}(y) = \frac{n-r+1}{r-1} f_{n,r-1}(y) \quad ,$$

and

$$(4.15) \quad \frac{e^{-2y}}{(1-e^{-y})^2} f_{n,r}(y) = \frac{(n-r+1)(n-r+2)}{(r-1)(r-2)} f_{n,r-2}(y)$$

and applying the formulae

$$(4.16)\qquad \int_0^\infty y\, f_{n,j}(y)\, dy = \sum_{i=1}^{j} \frac{1}{u-i+1}\ ,$$

$$\int_0^\infty y^2\, f_{n,j}(y)\, dy = \sum_{i=1}^{j} \frac{1}{(n-i+1)^2} + \left(\sum_{i=1}^{j} \frac{1}{n-i+1}\right)^2\ ,$$

we obtain after some manipulations

$$(4.17)\qquad Q = 1 + 2\,\frac{n-r+1}{n-r+2} \sum_{i=1}^{r-2} \frac{1}{n-i+1} + 2\,\frac{n-r+1}{(n-r+2)^2}$$

$$+ \frac{n(n-r+1)}{r-2}\left[\sum_{i=1}^{r-2} \frac{1}{(n-i+1)^2} + \left(\sum_{i=1}^{r-2} \frac{1}{n-i+1}\right)^2\right]$$

As shown in the present section, all the conditions for the consistency and asymptotic normality of the MLE, $\hat{\theta}_m$, are satisfied (see Lehmann [4]). Thus, for a fixed θ,

$$(4.18)\qquad \hat{\theta}_m - \theta = o_p\left(\frac{1}{\sqrt{m}}\right)\ ,\ \text{as}\ m\to\infty$$

and

$$(4.19)\qquad \sqrt{m}\ (\hat{\theta}_m - \theta) \xrightarrow{d} N\left(0\ ,\ \frac{\theta^2}{Q\left(\frac{t_o}{\theta}\right)}\right)$$

The convergence, however, is not uniform. For values of θ close to t_o the probability $P_\theta\{K_m = 0\}$ could be close to 1. For example, if $\theta = t_o$, $n=30$, $r=27$, $B(26;\ 30,\ 1-e^{-1}) = .999$.

Thus, in a sample of size $m=50$, $P\{K_m = 0\} = (.999)^{50} = .9512$. Thus, the probability is over .95 that the MLE would not exist. But, in a sample of size $m = 1{,}000$ this probability drops to .368.

5. Some Numerical Comparisons of the MLE and the MEE

In the present section we numerically illustrate the asymptotic variances of the MLE and MEE, and provide some simulation results. These simulations demonstrate the actual behavior of these estimators in samples of size m=20,30,40 and 50. The numerical computations of the present section are restricted to the case of n=30 and r=27 . These are the parameters used by Epstein et al [2] in the photodynamic bioassays.

The MLE estimator $\hat{\theta}_m$ and the MEE estimator $\hat{\theta}(\bar{X}_m^*)$ are both scale equivariant in the sense that, if $X_{n,r} \to \beta X_{n,r}$ for some $\beta > 0$, and correspondingly $t_o \to \beta t_o$ then $\hat{\theta}_m \to \beta \hat{\theta}_m$ and $\hat{\theta}(\beta\bar{X}_m^*) = \beta \hat{\theta}(\bar{X}_m^*)$. Moreover, since $\rho = t_o/\theta$ is invariant under such transformations, the (asymptotic) variances of $\hat{\theta}_m$ and $\hat{\theta}(\bar{X}_m^*)$ under θ are equal to θ^2 times the corresponding variances under $\theta = 1$. We therefore report here the variances of the estimators under the case of $\theta = 1$ and different values of $\rho = t_o/\theta$. In Table 5.1 we provide values of $Q(\rho)$ for some values of ρ . This function was determined according to (4.11) for $0 < \rho < \infty$ and according to (4.17) for $\rho = \infty$ (uncensored). Notice that $Q(\rho)$ is the value of the Fisher information function at $\theta = 1$, i.e., $I(1;\rho) = Q(\rho)$.

Table 5.1

Values of $Q(\theta)$ for n=30, r=27

ρ	1.80	2.00	2.25	3.00	4.50	∞
$Q(\rho)$	10.6275	13.7234	16.3008	18.6408	18.8340	18.8344

As seen in Table 5.1, the amount of information per observation is maximal when there is no censoring ($\rho=\infty$) . Moreover, if $\rho \geq 3$ the amount of information is close to the maximal one. The asymptotic efficiency of the MLE, under ρ , compared to the no-censoring case is

$$AE(\rho) = Q(\rho)/Q \tag{5.1}$$

Thus, as seen in Table 5.1, $AE(1.8) = .564$. On the other hand, $AE(3) = .99$. Thus, for θ values close to the censoring point t_o we need considerably larger samples, than in the uncensored case, to attain a specified asymptotic precision.

In Table 5.2 we present the values of the asymptotic variances of the MEE and the MLE, which have been determined according to formulae (2.16) and (4.19), respectively. The asymptotic relative efficiency (ARE)

Table 5.2

The Asymptotic Variances of the MEE and the MLE [in units of 10^{-3}], for $\theta = 1$, $m=50$ $n=30$, $r=27$ and the ARE

ρ	MEE	MLE	ARE
1.80	2.2723	1.8819	.828
2.00	1.6549	1.4574	.881
2.25	1.3153	1.2269	.933
2.57	1.1450	1.1152	.974
3.00	1.0809	1.0729	.993

of the MEE relative to the MLE is defined as the ratio of the asymptotic variances, i.e.,

$$ARE(\rho) = \frac{AV\{\hat{\theta}_m;\ 1,\ \rho\}}{AV\{\hat{\theta}(\bar{X}_m^*);\ 1,\ \rho\}} \tag{5.2}$$

As seen in Table 5.2, as ρ increases the value of the ARE approaches one. In other words, the asymptotic efficiency of the MEE is close to that of the MLE when the value of θ is considerably smaller than that of t_o, i.e., $\rho \geq 3$. If we suspect that θ is close to t_o the MLE is a more precise estimator than the MEE.

It is also interesting to study how the two estimators perform when the samples are not very large. For this purpose we have conducted a series of simulations. In these simulations we generated random samples of m values of $X^*_{n,r}$, for $n = 30$, $r = 27$, $t_o = 90$ and $\theta = 30$, 35, 40, 45 and 50. For the sample size, m, we considered the values $m = 20$, 30, 40 and 50. For each of these samples we computed the values of the MEE, $\hat{\theta}(\bar{X}^*_m)$, and of the MLE, $\hat{\theta}_m$. These trials were repeated M=500 times independently. If all the values of $X^*_{n,r}$ were censored, a message "NO-EST." was printed. It is interesting to notice that in the case of m=20 and θ=50 (ρ=1.8) we obtained this message two samples out of 500 independent samples. This conforms to the theory, since in the case of ρ=1.8, the probability of censoring is $B(26; 30, .8347) = .7543$. Thus, the probability that all m=20 observations will be censored is $(.7543)^{20} = .0036$. If such samples are simulated independently M=500 times, the expected number of samples with complete censoring is 1.8.

In Table 5.3 we present the means and mean square-errors of the simulated values of the MEE and MLE. It is interesting that in all cases the bias is negligible. In addition, the MSE's of the MEE are not significantly larger than those of the MLE when $\theta \leq 40$ ($\rho \geq 2.25$). Furthermore, when m=50 the MSE's of the MLE and of the MEE are not significantly larger than the corresponding asymptotic variances, even if $\theta > 40$ ($\rho < 2.25$). For example, when m=50, θ=50, ρ=1.8 and from Table 5.2, the asymptotic variance of the MLE is $2.5 \times 1.8819 = 4.705$.

Table 5.3

Means and MSE of M=500 simulated values of the MEE and MLE, for n=30, r=27, t_o=90, θ=30(5)50, and m=20(10)50

M	θ	MLE		MEE	
		MEAN	MSE	MEAN	MSE
20	30	30.25	2.358	29.97	2.319
	35	35.25	3.445	35.02	3.606
	40	40.36	5.475	40.11	5.789
	45	45.49	8.444	45.30	9.795
	50*	50.54	18.938	50.57	23.680
30	30	30.21	1.779	29.94	1.715
	35	35.21	2.520	34.96	2.577
	40	40.28	3.854	40.03	4.082
	45	45.37	5.824	45.16	6.537
	50	50.50	9.400	50.47	12.944
40	30	30.27	1.325	29.98	1.223
	35	35.28	1.866	35.00	1.793
	40	40.30	2.642	40.03	2.693
	45	45.33	3.943	45.10	4.470
	50	50.41	6.972	50.33	8.586
50	30	30.29	1.044	30.02	0.948
	35	35.31	1.457	35.02	1.364
	40	40.30	2.077	40.04	2.082
	45	45.35	3.101	45.10	3.382
	50	50.43	5.240	50.27	6.023

*The estimates are based on M′=498 samples.
Two samples were completely censored.

The corresponding standard-error of the variance estimate is approximately $SE=4.705\ (2/500)^{1/2} = .298$. Thus, $4.705 + 2\ SE = 5.300$, which is above the MSE value of the MLE in Table 5.3 . Similarly, the MSE of the MEE, for $m=50$, $\theta=50$, is within the .95-confidence interval constructed around the corresponding asymptotic variance. This is not, however, the case, when $m=20$ and $\theta \geq 45$. The MSE's of the MLE and the MEE are significantly larger than the corresponding asymptotic variances. For $m=20$, $\theta=50$, the asymptotic variance of the MLE is 11.762 . On the other hand, the estimated MSE is 18.938 . This shows that the actual variances of the MLE and the MEE are considerably larger than the asymptotic variances if the samples are not sufficiently large and $\rho \leq 2.00$.

Acknowledgment:

The author is indebted to Professor V. Susarla for many stimulating discussions on the present study.

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